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THE FOUR BLOCK PROBLEM FOR DISTRIBUTED SYSTEMS

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1. INTRODUCTION

In this paper we will study the singular values of a certain operator, *the four block operator*, (to be defined below), which appears naturally in many engineering H^∞ control problems. Indeed following the framework of the monograph of Francis [7], almost all such robust design problems can be formulated in terms of the spectral properties of such an operator. This includes the problems of sensitivity and mixed sensitivity minimization, model-matching and certain tracking problems. Our techniques give the optimal solution to this problem valid for a large class of distributed systems. We refer the reader to [7], and the references therein for more details on this subject, and the excellent work done on the part of both the engineering and mathematical communities in its connection. Finally for some recent important results on a Ricatti approach (in the finite dimensional case) to this subject, we refer the reader to [2] and [3].

In order to state a precise mathematical problem, we will first need to set up some notation. Accordingly, let $w, f, g, h, m \in H^\infty$, where w, f, g, h are rational and m is nonconstant inner. (All of our Hardy spaces will be defined on the unit disc D in the standard way.)

Set

$$A := \begin{bmatrix} Pw(S) & f(S) \\ g(S) & h(S) \end{bmatrix}$$

where $S : H^2 \to H^2$ denotes the unilateral shift, and P denotes orthogonal projection from H^2 to $H(m) := H^2 \ominus mH^2$. Note that if we let T denote the compression of S to $H(m)$, then $w(T)P = Pw(S)$. The *4-block problem* amounts to computing the norm and singular values of A. The point of this paper is to give a determinantal formula (see Theorem 1) for making this computation.

We should note that the techniques given here were heavily influenced by (and are based on) the previous work in [1], [4], [6], [12]. In [11], these ideas were applied to the mixed sensitivity ("two block") problem. (See also the closely related work in [8].) The present treatment essentially follows that of [1], [4], and [6].

It is very important to add that the skew Toeplitz framework developed in [1] also leads to the complete determination of the singular values of operators of the form A when the four blocks are taken to be **matrix-valued.** (See [1] for the precise definition and a more complete discussion on skew Toeplitz operators.) In this case the first block would correspond to a block Hankel operator, while the other blocks would correspond to matrix-valued Toeplitz operators. Since it is this kind of problem that one encounters in the H^∞ design of multiple input/multiple output systems, results on the norm of the matrix version of the four block operator should also be of interest even for finite dimensional systems. This is the topic of [5] to which we refer the interested reader. Moreover, all the proofs for the various results stated in this note can be found in [5].

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2. PRELIMINARY RESULTS AND NOTATION

In this section we make some remarks, and prove a result which will allow us to give the determinantal formula for the singular values and vectors of A in Section 3. We are basically following the line of argument given in [1], [4], and [6]. Thus we must first identify the essential norm of A (denoted by $\|A\|_e$), and then give an algorithmic procedure for determining a singular value of A, $\rho > \|A\|_e$. We are using the standard notation from operator theory as, for example, given in [9], [10]. In particular σ_e will denote the essential spectrum. We begin with the following result whose proof may be found in [5]:

PROPOSITION 1. *Notation as in the Introduction. Set*

$$\alpha := \max\left\{\left\| \begin{bmatrix} w(\zeta) & f(\zeta) \\ g(\zeta) & h(\zeta) \end{bmatrix} \right\| : \zeta \in \sigma_e(T)\right\} \tag{1}$$

$$\beta := \max\left\{\left\| \begin{bmatrix} 0 & f(\zeta) \\ g(\zeta) & h(\zeta) \end{bmatrix} \right\| : \zeta \in \partial D\right\} \tag{2}$$

where the norms in (1) and (2) are those in $L(\mathbf{C}^2)$. Then

$$\|A\|_e = \max(\alpha, \beta). \tag{3}$$

We will see in the next section how Proposition 1 leads to an algorithm for finding $\|A\|$.

3. ALGORITHM FOR NORM OF FOUR BLOCK OPERATOR

In this section we will discuss an algorithm for finding the singular values of A, which we will implement via a determinantal formula in Section 4. Again the line of argument we use here follows very closely that of [1], [4], and [6]. Using the notation of Section 2, we let $\rho > \max(\alpha, \beta)$. Notice that if $\|A\| > \|A\|_e$, then $\|A\|^2$ is an eigenvalue of A^*A.

So we begin by writing $w = a/k$, $f = b/k$, $g = c/k$, $h = d/k$, where a, b, c, d, k are polynomials of degree $\le n$. Then ρ^2 is an eigenvalue of A^*A if and only if

$$\begin{bmatrix} a(S)^*P & c(S)^* \\ b(S)^* & d(S)^* \end{bmatrix} \begin{bmatrix} Pa(S) & b(S) \\ c(S) & d(S) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \rho^2 k(S)^*k(S) & 0 \\ 0 & \rho^2 k(S)^*k(S) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{4}$$

for some non-zero

$$\begin{bmatrix} x \\ y \end{bmatrix} \in H^2 \oplus H^2.$$

Next for any polynomial p of degree $\le n$, we set

$$\hat{p}(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$$

for $z \in \mathbf{C}$, $z \ne 0$. With this notation if we multiply (4) by ζ^n, $\zeta \in \partial D$, we see that

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$$\begin{bmatrix} \hat{a}(S)P & \hat{c}(S) \\ \hat{b}(S) & \hat{d}(S) \end{bmatrix} \begin{bmatrix} Pa(S) & b(S) \\ c(S) & d(S) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \rho^2 \hat{k}(S)k(S) & 0 \\ 0 & \rho^2 \hat{k}(S)k(S) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \zeta^{n-1} + \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \zeta^{n-2} + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix} \tag{5}$$

for some

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix} \in \mathbf{C}^2$$

$j=1, \cdots, n$. Then applying P to (5), we obtain

$$\left\{ \begin{bmatrix} \hat{a}(T) & \hat{c}(T) \\ \hat{b}(T) & \hat{d}(T) \end{bmatrix} \begin{bmatrix} a(T) & b(T) \\ c(T) & d(T) \end{bmatrix} - \begin{bmatrix} \rho^2 \hat{k}(T)k(T) & 0 \\ 0 & \rho^2 \hat{k}(T)k(T) \end{bmatrix} \right\} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} T^{n-1} P 1 + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix} P 1 \tag{6}$$

where

$$\begin{bmatrix} p \\ q \end{bmatrix} := \begin{bmatrix} Px \\ Py \end{bmatrix}.$$

Next, we note that if $Px = Py = 0$, then $x = mx'$, $y = my'$ with $x', y' \in H^2$, and (5) becomes

$$\begin{bmatrix} 0 & \hat{c}(S) \\ \hat{b}(S) & \hat{d}(S) \end{bmatrix} \begin{bmatrix} 0 & b(S) \\ c(S) & d(S) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} \rho^2 \hat{k}(S)k(S) & 0 \\ 0 & \rho^2 \hat{k}(S)k(S) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \overline{m(\zeta)}\, \zeta^{n-1} + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix} \overline{m(\zeta)}. \tag{7}$$

Multiplying (7) by ζ^{-n}, and applying the orthogonal projection onto $H^2 \oplus H^2$, it is easy to see that

$$\left\| \begin{bmatrix} 0 & b(S) \\ c(S) & d(S) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \right\|^2 = \rho^2 \left\| \begin{bmatrix} k(S)x' \\ k(S)y' \end{bmatrix} \right\|^2$$

and hence,

$$\left\| \begin{bmatrix} 0 & b(S) \\ c(S) & d(S) \end{bmatrix} \begin{bmatrix} x'' \\ y'' \end{bmatrix} \right\| = \rho \left\| \begin{bmatrix} x'' \\ y'' \end{bmatrix} \right\|$$

where $x'' := k(S)x'$, and $y'' := k(S)y'$. Since

$$\left\| \begin{bmatrix} 0 & b(S) \\ c(S) & d(S) \end{bmatrix} \right\| \le \beta < \rho.$$

we deduce that $x'' = y'' = 0$, and thus $x = y = 0$.

Now let us assume that

$$\begin{bmatrix} p \\ q \end{bmatrix} \in H(m) \oplus H(m)$$

satisfies (6) and that

$$\begin{bmatrix} p \\ q \end{bmatrix} \ne 0.$$

Then

$$\left\{ \begin{bmatrix} \hat{a}(S)P & \hat{c}(S) \\ \hat{b}(S) & \hat{d}(S) \end{bmatrix} \begin{bmatrix} Pa(S) & b(S) \\ c(S) & d(S) \end{bmatrix} - \begin{bmatrix} \rho^2 \hat{k}(S)k(S) & 0 \\ 0 & \rho^2 \hat{k}(S)k(S) \end{bmatrix} \right\} \begin{bmatrix} p \\ q \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \zeta^{n-1} + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix} + m \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

for some $\xi, \eta \in H^2$.

But (5) and (8) can be re-written as

$$\begin{bmatrix} \hat{a}(S)a(T) & \hat{a}(S)b(T) \\ \hat{b}(S)a(T) & 0 \end{bmatrix} \begin{bmatrix} Px \\ Py \end{bmatrix} + B(S) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \zeta^{n-1} + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \tag{5a}$$

respectively,

$$\begin{bmatrix} \hat{a}(S)a(T) & \hat{a}(S)b(T) \\ \hat{b}(S)a(T) & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + B(S) \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \zeta^{n-1} + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix} + m \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{8a}$$

where

$$B(\zeta) := \begin{bmatrix} 0 & \hat{c}(\zeta) \\ \hat{b}(\zeta) & \hat{d}(\zeta) \end{bmatrix} \begin{bmatrix} 0 & b(\zeta) \\ c(\zeta) & d(\zeta) \end{bmatrix} - \begin{bmatrix} \rho^2(\hat{k}\, k)(\zeta) & 0 \\ 0 & \rho^2(\hat{k}\, k)(\zeta) \end{bmatrix} \tag{9}$$

for $\zeta \in \mathbf{C}$. Obviously, (5a) and (8a) coincide provided

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = B(S) \begin{bmatrix} \xi' \\ \eta' \end{bmatrix} \tag{10}$$

for some $\xi', \eta' \in H^2$.

We can now summarize the above discussion with the following:

PROPOSITION 2. *There exists an eigenvector*

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \in H^2 \oplus H^2, \; z \ne 0$$

satisfying (4) (i.e., ρ is a singular value of A) if and only if there exists a non-zero

$$\begin{bmatrix} p \\ q \end{bmatrix} \in H(m) \oplus H(m)$$

satisfying (8a) and (10).

In the next section we will construct the determinant of the linear system associated to (8a) and (10).

4. DETERMINANTAL FORMULA

In this section, we will give the explicit determinantal formula for the singular values of the four block operator A. We use the previous notation here. Moreover, by slight abuse of notation, ζ will denote a complex variable as well as an element of ∂D (the unit circle). The context will always make the meaning clear. Of course, when $\zeta \in \partial D$, then $\overline{\zeta} = 1/\zeta$.

As in Section 3, we let $\rho > \max(\alpha, \beta)$, and as we have seen from Proposition 2, we are looking for a non-zero

$$\begin{bmatrix} p \\ q \end{bmatrix} \in H(m) \oplus H(m)$$

and vectors

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix} \in \mathbf{C}^2,$$

$$\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} \in H^2 \oplus H^2$$

satisfying

$$\begin{bmatrix} \hat{a}(S)a(T) & \hat{a}(S)b(T) \\ \hat{b}(S)a(T) & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + B \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \zeta^{n-1} + \cdots + \begin{bmatrix} u_n \\ v_n \end{bmatrix} + mB \begin{bmatrix} \xi' \\ \eta' \end{bmatrix}. \tag{11}$$

We now introduce the matrix-valued polynomial operator

$$C := \begin{bmatrix} \hat{a}a & \hat{a}b \\ \hat{b}a & 0 \end{bmatrix} + B \tag{12}$$

Noting that $\overline{m}p, \overline{m}q \in L^2 \ominus H^2$, we can express

$$\begin{aligned} m &= m_0 + m_1\zeta + \ldots.. \\ \overline{m}p &= p_{-1}\zeta^{-1} + p_{-2}\zeta^{-2} + \cdots \\ \overline{m}q &= q_{-1}\zeta^{-1} + q_{-2}\zeta^{-2} + \cdots \end{aligned} \tag{13}$$

and write

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$$C = \sum_{l=0}^{2n} C_l \zeta^l, \ B = \sum_{l=0}^{2n} B_l \zeta^l, \ a = \sum_{l=0}^{n} a_l \zeta^l, \ b = \sum_{l=0}^{n} b_l \zeta^l. \qquad (14)$$

Next we let P_{H^2} denote orthogonal projection from L^2 to H^2. Then

$$a(T)p = ap - m\ P_{H^2}(a\overline{m}p)$$

$$b(T)q = bq - m\ P_{H^2}(b\overline{m}q)$$

and thus (11) is equivalent to

$$C\begin{bmatrix} p \\ q \end{bmatrix} - \sum_{j=1}^{n} \zeta^{n-j} \begin{bmatrix} u_j \\ v_j \end{bmatrix} = m\left\{ \begin{bmatrix} \hat{a} & \hat{a} \\ \hat{b} & 0 \end{bmatrix} \begin{bmatrix} P_{H^2}(a\overline{m}p) \\ P_{H^2}(b\overline{m}q) \end{bmatrix} + B \begin{bmatrix} \xi' \\ \eta' \end{bmatrix} \right\} =: m \begin{bmatrix} \xi'' \\ \eta'' \end{bmatrix}. \qquad (15)$$

Now it is well-known (and easy to compute) that

$$T^j p = \zeta^j p - m \sum_{i=1}^{j} \zeta^{j-i} p_{-i}$$

and similarly, of course, for q and $P\,1 = 1 - m\ \overline{m(0)}$. (Recall that $P : H^2 \to H(m)$ denotes orthogonal projection.) By applying P to (15), we get that

$$C(T)\begin{bmatrix} p \\ q \end{bmatrix} - \sum_{j=1}^{n} T^{n-j} P\, 1 \begin{bmatrix} u_j \\ v_j \end{bmatrix} = 0. \qquad (16)$$

Notice that this means

$$C\begin{bmatrix} p \\ q \end{bmatrix} - \sum_{j=1}^{n} \zeta^{n-j} \begin{bmatrix} u_j \\ v_j \end{bmatrix} = mP_{H^2}\left\{ C\begin{bmatrix} \overline{m}p \\ \overline{m}q \end{bmatrix} - \sum_{j=1}^{n} \zeta^{n-j}\overline{m} \begin{bmatrix} u_j \\ v_j \end{bmatrix} \right\}. \qquad (16a)$$

Consequently, we see that (11) is equivalent to (16a) and

$$P_{H^2} C \begin{bmatrix} \overline{m}p \\ \overline{m}q \end{bmatrix} - \sum_{j=1}^{n} P_{H^2}(\zeta^{n-j}\overline{m}) \begin{bmatrix} u_j \\ v_j \end{bmatrix} = \begin{bmatrix} \xi'' \\ \eta'' \end{bmatrix}. \qquad (16b)$$

Now

$$P_{H^2} C \begin{bmatrix} \overline{m}p \\ \overline{m}q \end{bmatrix} = P_{H^2}\left\{ \sum_{l=0}^{2n} C_l \zeta^l \sum_{j=1}^{\infty} \zeta^{-j} \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix} \right\} = \sum_{s=0}^{2n-1} \zeta^s \sum_{j=1}^{2n-s} C_{s+j} \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix}. \qquad (17a)$$

Similarly, we have

$$P_{H^2} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \overline{m}p \\ \overline{m}q \end{bmatrix} = \sum_{s=0}^{n-1} \zeta^s \sum_{j=1}^{n-s} \begin{bmatrix} a_{s+j} & 0 \\ 0 & b_{s+j} \end{bmatrix} \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix}, \qquad (17b)$$

and

$$P_{H^2} \sum_{j=1}^{n} (\zeta^{n-j}\overline{m}) \begin{bmatrix} u_j \\ v_j \end{bmatrix} = \sum_{j=0}^{n-1} \zeta^s \sum_{j=1}^{n-s} \overline{m}_{n-s-j} \begin{bmatrix} u_j \\ v_j \end{bmatrix}. \qquad (17c)$$

Thus (16a) and (16b) are equivalent to

$$C\begin{bmatrix} p \\ q \end{bmatrix} - \sum_{j=1}^{n} \zeta^{n-j} \begin{bmatrix} u_j \\ v_j \end{bmatrix} = m\left\{ \sum_{s=0}^{2n-1} \zeta^s \sum_{j=1}^{2n-s} C_{s+j} \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix} - \sum_{s=0}^{n-1} \zeta^s \sum_{j=1}^{n-s} \overline{m}_{n-s-j} \begin{bmatrix} u_j \\ v_j \end{bmatrix} \right\} \qquad (16c)$$

and

$$B\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = \sum_{s=0}^{2n-1} \zeta^s \sum_{j=1}^{2n-s} C_{s+j} \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix} - \sum_{l=0}^{n} \sum_{s=0}^{n-1} \zeta^{s+l} \sum_{j=1}^{n-s} \left\{ \begin{bmatrix} \overline{a}_{n-l} a_{s+j} & \overline{a}_{n-l} b_{s+j} \\ \overline{b}_{n-l} a_{s+j} & 0 \end{bmatrix} \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix} + \overline{m}_{n-s-j} \begin{bmatrix} u_j \\ v_j \end{bmatrix} \right\}. \qquad (16d)$$

Taking a respite (!) from all of these computations, let us summarize the above discussion with:

PROPOSITION 3. *Equality (11) is equivalent to the two equalities (16c) and (16d). The eigenvalue equation (4) with*

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$$

is thus equivalent (see Proposition 2) to (16c), (16d), and

$$\begin{bmatrix} p \\ q \end{bmatrix} \neq 0. \qquad (18)$$

Next set

$$d_c(\zeta) := \det C(\zeta) \qquad (19)$$

and

$$C^+(\zeta) = \sum_{j=0}^{2n} C_j^+ \zeta^j$$

where

$$C_j = \begin{bmatrix} c_{j,11} & c_{j,12} \\ c_{j,21} & c_{j,22} \end{bmatrix}$$

$$C_j^+ = \begin{bmatrix} c_{j,22} & -c_{j,12} \\ -c_{j,21} & c_{j,11} \end{bmatrix}$$

for $0 \le j \le 2n$. Note that $C^+(\zeta)C(\zeta) = d_c(\zeta)I$.

We will now make a technical assumption in order to simplify our exposition. Below we will discuss how to remove this *assumption of genericity*. Explicitly, we assume that

$$d_c \text{ has distinct roots all of which are non-zero.} \qquad (20)$$

We now have the following result (see [5] for the proof):

LEMMA 1. *Under assumption (20), $d_c(\zeta)$ has r zeros $\alpha_1, ..., \alpha_r \in D$, r zeros $1/\overline{\alpha}_1, \cdots, 1/\overline{\alpha}_r \notin \overline{D}$, and $4n - 2r$ zeros $\alpha_{2r+1}, \cdots, \alpha_{4n} \in \partial D \backslash \sigma(T)$.*

Continuing our computation, we note that by multiplying by $C^+(\zeta)$, we can express (16c) equivalently as

$$d_c \begin{bmatrix} p \\ q \end{bmatrix} = \sum_{j=1}^{n} (E_j + mF_j) \begin{bmatrix} u_j \\ v_j \end{bmatrix} + m \sum_{j=1}^{2n} G_j \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix} \qquad (22)$$

where the matrix-valued polynomials E_j, F_j, G_j are given by

$$E_j := \sum_{l=0}^{2n} \zeta^{j+l} C_l^+,$$

$$F_j := -\sum_{s=0}^{n-j} \sum_{l=0}^{2n} \zeta^{s+l} \overline{m}_{n-s-j} C_l^+ \qquad (22a)$$

$$G_j := \sum_{s=0}^{2n-j} \sum_{l=0}^{2n} \zeta^{s+l} C_l^+ C_{s+j}$$

for $1 \le j \le n$.

We are almost done now! Indeed arguing precisely as in [6], we note that since $p, q \in H(m)$, they must be analytic in a neighborhood of $\partial D \backslash \sigma(T)$ as well as in D. Hence using Lemma 1, from (22) we see that

$$\sum_{j=1}^{n} (E_j(\alpha_i) + m(\alpha_i)F_j(\alpha_i)) \begin{bmatrix} u_j \\ v_j \end{bmatrix} + m(\alpha_i) \sum_{j=1}^{2n} G_j(\alpha_i) \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix} = 0 \qquad (23a)$$

for $1 \le i \le r$, and for $2r+1 \le i \le 4n$ For $|\zeta|=1$, if we multiply (22) by $\overline{\zeta}^{4n}\overline{m}$, we get

$$\overline{\zeta}^{4n} d_c \begin{bmatrix} \overline{m}p \\ \overline{m}q \end{bmatrix} = \sum_{j=1}^{n} (\overline{m}\,\overline{\zeta}^{4n} E_j + \overline{\zeta}^{4n} F_j) \begin{bmatrix} u_j \\ v_j \end{bmatrix} + \sum_{j=1}^{2n} \overline{\zeta}^{4n} G_j \begin{bmatrix} p_{-j} \\ q_{-j} \end{bmatrix}. \qquad (24)$$

But this last equation admits an analytic extension to the complement of the disc, i.e. all the functions are analytic in $1/\zeta$. Set

$$E_j^o(1/\zeta) := (1/\zeta)^{4n} E_j(\zeta)$$

and similarly for d_c^o, F_j^o, and G_j^o. Hence we can express (24) equivalently as

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$$d_c^o(1/\zeta)\begin{bmatrix}(\overline{m}p)(1/\zeta)\\(\overline{m}q)(1/\zeta)\end{bmatrix} = \sum_{j=1}^{n}(\overline{m(1/\overline{\zeta})}E_j^o(1/\zeta) + F_j^o(1/\zeta))\begin{bmatrix}u_j\\v_j\end{bmatrix} + \sum_{j=1}^{2n}G_j^o(1/\zeta)\begin{bmatrix}p_{-j}\\q_{-j}\end{bmatrix}. \tag{24a}$$

Then from (24a) and Lemma 1, we get that

$$\sum_{j=1}^{n}(\overline{m(\alpha_i)}E_j^o(\overline{\alpha_i}) + F_j^o(\overline{\alpha_i}))\begin{bmatrix}u_j\\v_j\end{bmatrix} + \sum_{j=1}^{2n}G_j^o(\overline{\alpha_i})\begin{bmatrix}p_{-j}\\q_{-j}\end{bmatrix} = 0 \quad \textbf{(24b)}$$

for $1 \le i \le r$.

We now play the analogous game with the B operator that we just did with the C operator. Indeed, setting

$$d_b(\zeta) := \det B(\zeta)$$

we have the analogy of (21), and since $\rho > \beta$, we have that $d_b(\zeta) \neq 0$ for $\zeta \in \partial D$. Again we make the assumption of genericity that

d_b has distinct roots all of which are non-zero. (25)

Just like for lemma 1 above, we see that d_b has $2n$ zeros $\beta_1, \ldots, \beta_{2n} \in D$, and $2n$ zeros $1/\overline{\beta}_1, \cdots, 1/\overline{\beta}_{2n}$ in the complement of $\overline{D}$.

We now set (as above)

$$B^+(\zeta) := \sum_{j=0}^{2n}B_j^+\zeta^j$$

where

$$B_j = \begin{bmatrix}b_{j,11} & b_{j,12}\\ b_{j,21} & b_{j,22}\end{bmatrix}$$

$$B_j^+ = \begin{bmatrix}b_{j,22} & -b_{j,12}\\ -b_{j,21} & b_{j,11}\end{bmatrix}$$

for $0 \le j \le 2n$. Note that $B^+(\zeta)B(\zeta) = d_b(\zeta)$. Finally, since

$$\begin{bmatrix}\xi'\\ \eta'\end{bmatrix} \in H^2 \oplus H^2$$

(16d) implies for $1 \le i \le 2n$, the equations

$$\sum_{j=1}^{2n}H_j(\beta_i)\begin{bmatrix}p_{-j}\\q_{-j}\end{bmatrix} + \sum_{j=1}^{n}K_j(\beta_i)\begin{bmatrix}u_j\\v_j\end{bmatrix} = 0 \tag{26}$$

where

$$H_j(\zeta) := \sum_{s=0}^{2n-j}\sum_{l=0}^{2n}\zeta^{s+l}B_l^+C_{s+j} - \sum_{i=0}^{n}\sum_{s=0}^{n-j}\sum_{l=0}^{2n}\zeta^{s+l+i}B_l^+\begin{bmatrix}\overline{a}_{n-i}a_{s+j} & \overline{a}_{n-i}b_{s+j}\\ \overline{b}_{n-i}a_{s+j} & 0\end{bmatrix} \tag{26a}$$

$$K_j(\zeta) := -\sum_{s=0}^{n-j}\sum_{l=0}^{2n}\begin{bmatrix}\zeta^{s+l} & 0\\ 0 & \zeta^{s+l}\end{bmatrix}B_l^+\overline{m}_{n-s-j}.$$

We are at long last ready to state our main result (see [5] for the proof):

THEOREM 1. *Assume the genericity conditions (20) and (25) hold. Then ρ is a singular value of the four block operator A if and only if*

$$\det\begin{bmatrix}M_1 & M_3\\ M_4 & M_2\end{bmatrix} = 0$$

where

$$M_1 := \begin{bmatrix}
E_1(\alpha_1) + m(\alpha_1)F_1(\alpha_1) & . & E_n(\alpha_1) + m(\alpha_1)F_n(\alpha_1)\\
. & . & .\\
. & . & .\\
. & . & .\\
E_1(\alpha_r) + m(\alpha_r)F_1(\alpha_r) & . & E_n(\alpha_r) + m(\alpha_r)F_n(\alpha_r)\\
E_1(\alpha_{2r+1}) + m(\alpha_{2r+1})F_1(\alpha_{2r+1}) & . & E_n(\alpha_{2r+1}) + m(\alpha_{2r+1})F_n(\alpha_{2r+1})\\
. & . & .\\
. & . & .\\
. & . & .\\
E_1(\alpha_{4n}) + m(\alpha_{4n})F_1(\alpha_{4n}) & . & E_n(\alpha_{4n}) + m(\alpha_{4n})F_n(\alpha_{4n})\\
\overline{m(\alpha_1)}E_1^o(\overline{\alpha}_1) + F_1^o(\overline{\alpha}_1) & . & \overline{m(\alpha_1)}E_n^o(\overline{\alpha}_1) + F_n^o(\overline{\alpha}_1)\\
. & . & .\\
. & . & .\\
. & . & .\\
\overline{m(\alpha_r)}E_1^o(\overline{\alpha}_r) + F_1^o(\overline{\alpha}_r) & . & \overline{m(\alpha_r)}E_n^o(\overline{\alpha}_r) + F_n^o(\overline{\alpha}_r)
\end{bmatrix}$$

$$M_2 := \begin{bmatrix}K_1(\beta_1) & . & K_n(\beta_1)\\ . & . & .\\ . & . & .\\ . & . & .\\ K_1(\beta_{2n}) & . & K_n(\beta_{2n})\end{bmatrix}$$

$$M_3 := \begin{bmatrix}
m(\alpha_1)G_1(\alpha_1) & . & m(\alpha_1)G_{2n}(\alpha_1)\\
. & . & .\\
. & . & .\\
. & . & .\\
m(\alpha_r)G_1(\alpha_r) & . & m(\alpha_r)G_{2n}(\alpha_r)\\
m(\alpha_{2r+1})G_1(\alpha_{2r+1}) & . & m(\alpha_{2r+1})G_{2n}(\alpha_{2r+1})\\
. & . & .\\
. & . & .\\
. & . & .\\
m(\alpha_{4n})G_1(\alpha_{4n}) & . & m(\alpha_{4n})G_{2n}(\alpha_{4n})\\
G_1^o(\overline{\alpha}_1) & . & G_{2n}^o(\overline{\alpha}_1)\\
. & . & .\\
. & . & .\\
. & . & .\\
G_1^o(\overline{\alpha}_r) & . & G_{2n}^o(\overline{\alpha}_r)
\end{bmatrix}$$

$$M_4 := \begin{bmatrix}H_1(\beta_1) & . & H_{2n}(\beta_1)\\ . & . & .\\ . & . & .\\ . & . & .\\ H_1(\beta_{2n}) & . & H_{2n}(\beta_{2n})\end{bmatrix}.$$

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